Valid restricted randomization for small experiments

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Joint work with Josh Paik (Penn State University)

Introductory example

Suppose that scientists at a horticultural research institute are planning an experiment to compare three varieties of tomato, labelled *A*, *B* and *C*, to see which gives the biggest yield (in weight of fruit per plant). They propose to use a greenhouse which has room for nine tomato plants in a single row. The initial, systematic layout is shown below.

position ("plot")	1	2	3	4	5	6	7	8	9
variety ("treatment")	*A*	*A*	*A*	*B*	*B*	*B*	*C*	*C*	*C*

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Angel: Can we use a smaller set of potential layouts with the properties that

(a) we never get a series of 3 adjacent plots with the same variety;

(b) we do not get the bias mentioned by Devil 2?

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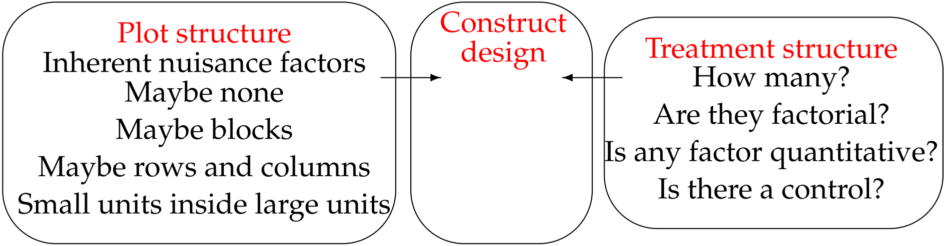
In 1939, Yates proposed the term restricted randomization for any method of randomization that does not include all possible layouts (but preferably avoids both forms of bias).

Terminology: UK, Australia

Plot structure
Inherent nuisance factors
Maybe none
Maybe blocks
Maybe rows and columns
Small units inside large units

Treatment structure
How many?
Are they factorial?
Is any factor quantitative?
Is there a control?

Terminology: UK, Australia



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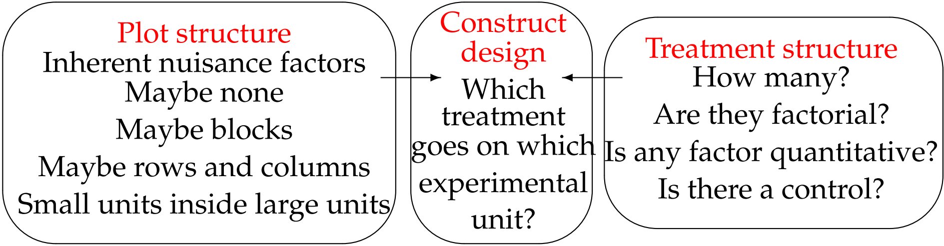

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Randomization

Plot structure
Inherent nuisance factors
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→

Construct design
Which treatment goes on which experimental unit?

←

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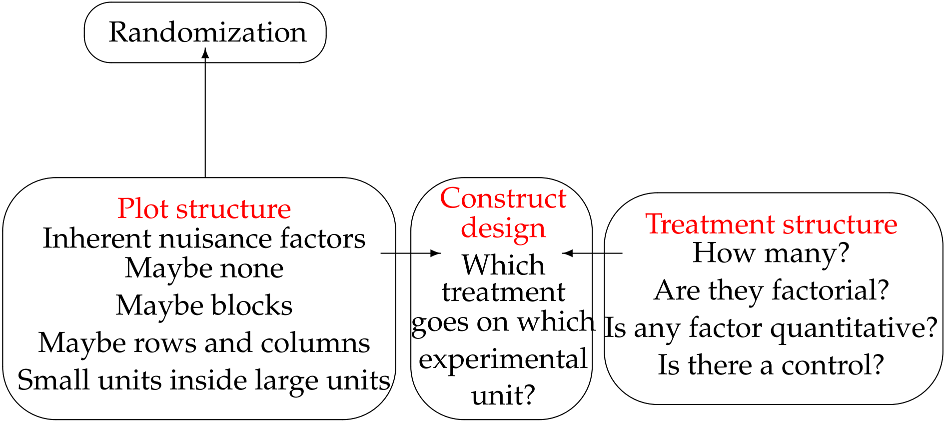

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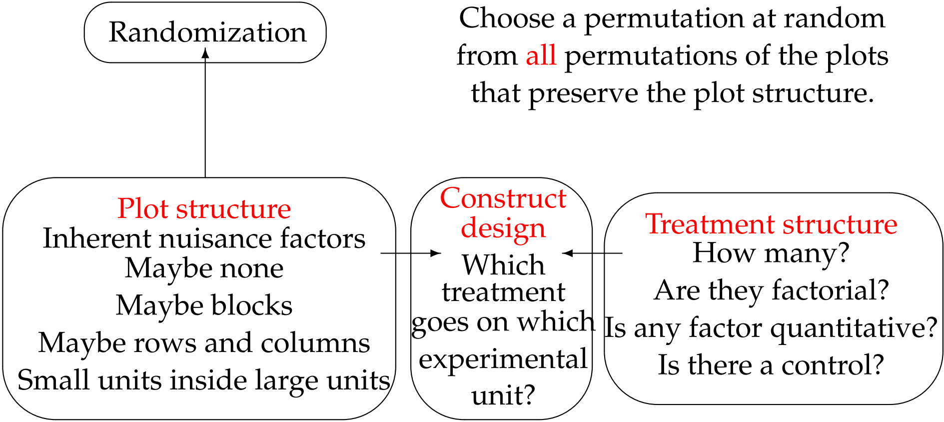

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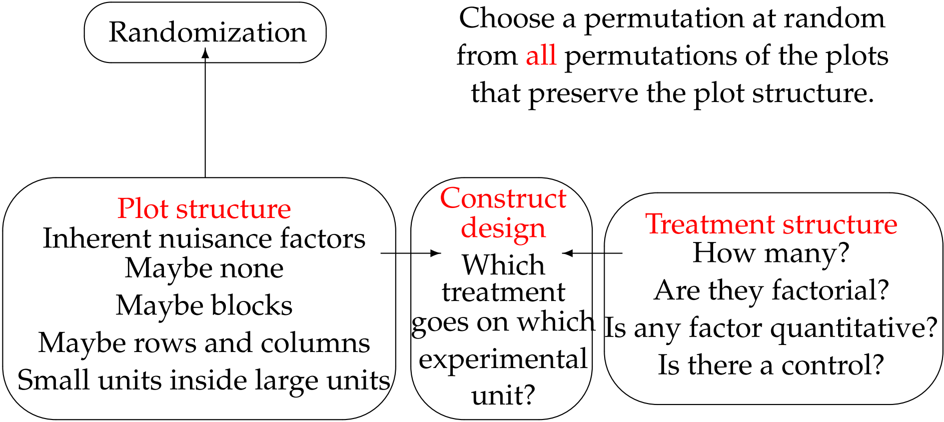

Restricted randomization means using only a proper subset of the possible layouts.

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From now on, I will continue to use the term restricted randomization in the sense that Yates did.

Valid randomization

In the context of experiments with a single error term in the analysis of variance,
Fisher and Yates said that a method of randomization should have the property that,
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Random choice of layout for the experiment turns all our statistical notions (such as estimators and mean squares) into random variables.

More technical details

In an unblocked experiment with equal replication, a method of randomization is strongly valid if there are probabilities p_1 and p_2 such that, whenever α and β are distinct plots,

$$P(T(\alpha) = T(\beta) = i) = p_1 \quad \text{for each treatment } i \tag{1}$$

and

$$P(T(\alpha) = i \text{ and } T(\beta) = j) = p_2 \quad \text{whenever } i \neq j. \tag{2}$$

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If a strongly valid method of randomization is used, then all pairs of distinct plots have the same probability p_2 of contributing to the estimator of the difference between any ordered pair of distinct treatments, and probability vp_1 of contributing to the mean square for error, where v is the number of treatments.

Even more technical details

Suppose that there are v treatments, each with replication r, so that the number N of plots is given by $N = vr$. If Equations (1) and (2) are satisfied, then

$$p_1 = \frac{1}{v}\frac{r-1}{N-1} \tag{3}$$

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For a Latin square, there is one such pair for plots in the same row or in the same column, and another such pair for plots which are in different rows and different columns.

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Here we restrict attention to unblocked experiments where the plots form a single line. Denote the design for such an unblocked experiment by Δ.

First strongly valid method, using permutation groups

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In 1976–1978, RAB was employed as a post-doc at the Agricultural Research Council Unit of Statistics (in Edinburgh) because her DPhil thesis was about finite permutation groups.
This led to a paper on restricted randomization for Latin squares (and other things) in *Biometrika* in 1983.

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After his death in 1971, Youden's widow and the IMS agreed to the publication of Youden's preprint in *Technometrics* in 1972.

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Given such a rectangle, Youden proposed randomizing by choosing one of the rows with equal probability and then randomizing the actual treatments to the letters in that row.

An example from Youden's paper

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There are $3 \times 2 = 6$ columns.

Each row has three letters, each occurring twice.

There is no "very bad" row, such as $AABBCC$.

Each pair of columns have the same letter in precisely one row.

An example from Youden's paper

Suppose that $v = 3$ and $r = 2$.

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Randomize by choosing one of the 5 rows with equal probability, then randomizing the 3 treatments to A, B and C.

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A	*B*	*C*	*B*	*A*	*C*
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So Γ has blocks of size r, and is resolved, with replication m.

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Likewise, $P(T(\alpha) \neq T(\beta)) = (m-\lambda)/m$, and this probability is equally split between the $v(v-1)$ ordered pairs of distinct treatments in Δ, and so

$$p_2 = \frac{1}{v(v-1)}\frac{m-\lambda}{m} = \frac{1}{v(v-1)}\frac{(N-r)}{N-1} = \frac{1}{v}\frac{r}{N-1},$$

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so that Equations (2) and (4) are satisfied.

As with the first method, the task now is to find a permutation of the columns of the $m \times N$ rectangle such none of the m rows gives a bad pattern.

A solution to the introductory example

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A	B	A	C	A	C	C	B	B
A	A	B	B	C	A	C	B	C
A	B	B	A	C	C	B	C	A
A	B	C	B	B	C	A	A	C

$\{1,3,5\}$ $\{2,8,9\}$ $\{4,6,7\}$
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A	*B*	*B*	*A*	*C*	*C*	*B*	*C*	*A*
A	*B*	*C*	*B*	*B*	*C*	*A*	*A*	*C*

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Each row contains exactly two adjacent pairs of columns with the same letter.

Moreover, no row has all three occurrences of any letter in either the first or last four columns.

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When $v = 2$ and $r = 3$, then Γ must be a resolved balanced incomplete-block design for 6 treatments in blocks of size 3.
The smallest such design consists of all triples of treatments, and so we cannot avoid the layout *AAABBB*.

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For example, here is a valid restricted randomization scheme for $v = 5$ and $r = 3$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A	*C*	*E*	*B*	*E*	*B*	*D*	*D*	*A*	*E*	*C*	*C*	*D*	*A*	*B*
A	*A*	*C*	*B*	*D*	*C*	*D*	*A*	*B*	*B*	*E*	*D*	*E*	*C*	*E*
B	*A*	*D*	*B*	*A*	*E*	*E*	*C*	*E*	*C*	*C*	*D*	*B*	*A*	*D*
E	*B*	*D*	*B*	*C*	*C*	*B*	*A*	*D*	*E*	*D*	*E*	*C*	*A*	*A*
D	*E*	*E*	*B*	*A*	*D*	*C*	*A*	*A*	*C*	*D*	*B*	*E*	*B*	*C*
A	*D*	*B*	*B*	*A*	*D*	*E*	*B*	*C*	*D*	*E*	*C*	*C*	*E*	*A*
C	*A*	*C*	*B*	*B*	*E*	*C*	*E*	*A*	*D*	*B*	*E*	*D*	*D*	*A*

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A	A	C	B	D	C	D	A	B	B	E	D	E	C	E
B	A	D	B	A	E	E	C	E	C	C	D	B	A	D
E	B	D	B	C	C	B	A	D	E	D	E	C	A	A
D	E	E	B	A	D	C	A	A	C	D	B	E	B	C
A	D	B	B	A	D	E	B	C	D	E	C	C	E	A
C	A	C	B	B	E	C	E	A	D	B	E	D	D	A

Two identical duads per row.

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A	*A*	*C*	*B*	*D*	*C*	*D*	*A*	*B*	*B*	*E*	*D*	*E*	*C*	*E*
B	*A*	*D*	*B*	*A*	*E*	*E*	*C*	*E*	*C*	*C*	*D*	*B*	*A*	*D*
E	*B*	*D*	*B*	*C*	*C*	*B*	*A*	*D*	*E*	*D*	*E*	*C*	*A*	*A*
D	*E*	*E*	*B*	*A*	*D*	*C*	*A*	*A*	*C*	*D*	*B*	*E*	*B*	*C*
A	*D*	*B*	*B*	*A*	*D*	*E*	*B*	*C*	*D*	*E*	*C*	*C*	*E*	*A*
C	*A*	*C*	*B*	*B*	*E*	*C*	*E*	*A*	*D*	*B*	*E*	*D*	*D*	*A*

Two identical duads per row. No identical triads in any row.