A-optimal designs for state estimation in networks

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Mathematical Statistics

mODa 13 – Model-Oriented Data Analysis and Optimum Design

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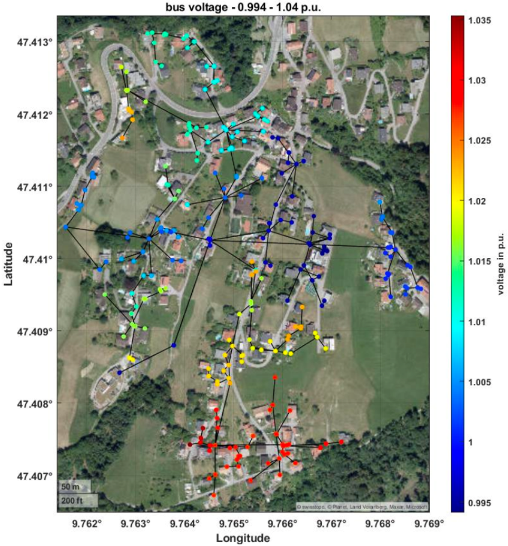
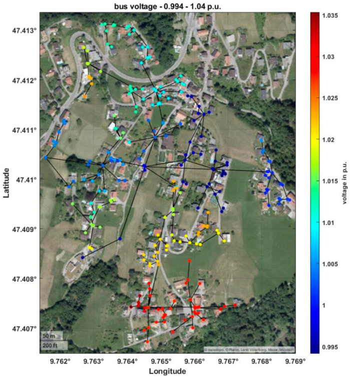
Motivating Example I



Motivating Example II

Situation:

- Electrial power distribution grids (PDG) of medium and low-voltage levels.
- Cooperating electrical engineers study the state of these grids.
- Different households are connected.



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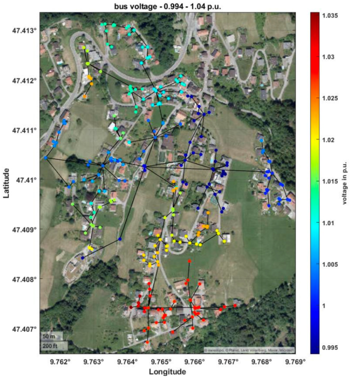


Target:

1. Appropriate statistical modelling of the random states in these grids.
2. Optimal positioning of measurement devices of different precision under budget constraints.

Motivating Example III

Specific problem:

1. Different types of measurements can be placed at each node.
2. Due to costs, the most precise measurement device cannot be set up at all nodes.

Motivating Example III

Specific problem:

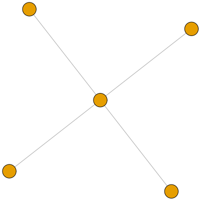
1. Different types of measurements can be placed at each node.
2. Due to costs, the most precise measurement device cannot be set up at all nodes. Pseudo measurements have to be used at some nodes.
3. At which nodes is it sufficient to use pseudo measurements?

Motivating Example IV

Ideas:

1. Use methods from graph theory for the model formulation.
2. Try to find some analytical solutions for optimal designs.

Motivating Example IV



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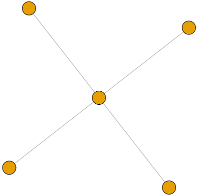
Observing states of a network in PDG – The star-network

- The network consists of $I + 1$ nodes $0, \ldots, I$
- The unknown expected states are given by: $s = (s_0, \ldots, s_I)^\top$



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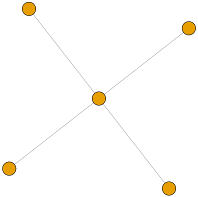

- Let a be the influence of the state s_i on the expected observation,
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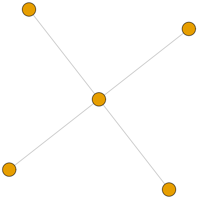

- The expected observations are then:

$$
\begin{aligned}
Y_0 &= as_0 + \sum_{i=1}^{I} bs_i, \\
Y_i &= as_i + bs_0\,, \quad i = 1, \ldots, I\,.
\end{aligned}
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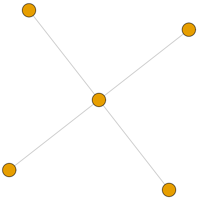
$$Y=(Y_0,\ldots,Y_I)^\top=\begin{pmatrix} a & b1_I^\top \\ b1_I & a\mathbb{I}_{I\times I}\end{pmatrix}s$$



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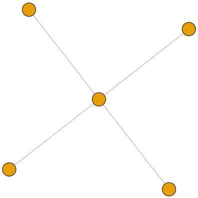

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Observing states of a network in PDG – General network

Assumptions:

- The PDG is well described by an undirected graph.
- The *known* weights of the edges describe the influence of adjacent nodes' states on the observation of a particular node.
  - $\Rightarrow$ These weights are stored in the adjacency matrix $\mathbb{A}$.
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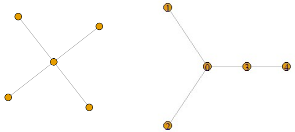

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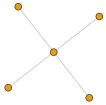

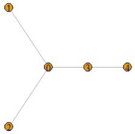

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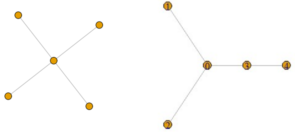
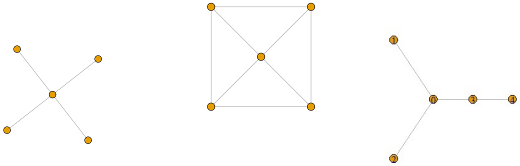


Leaving the idealistic world

The expected observations are: $Y = \mathbb{X}s$

We now integrate two types of errors into our model:

1. the expected state vector s is not passing into the model directly, but with some random error.
2. the observations taken at each node of the network are more or less noisy (due to the different types of measurements allocated).



A network model for state estimation I

Simultaneous observations of the complete network at N time points:

$$\begin{aligned} Y_n &= \mathbb{X} S_n + E_n\,, \quad n = 1, \dots, N \\ S_n &= s + Z_n \end{aligned}$$

- Y_n is the $(I+1)$-dimensional vector
- $\mathbb{X}$ is the known influence matrix of the network

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- S_n $(I+1)$-dimensional random state at time point n
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  - $Z_1, \ldots, Z_N$ are i.i.d., $\mathbb{E}[Z_n] = 0$, $\mathrm{Cov}(Z_n) = \sigma^2 \mathbb{D}_Z$

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where $\sigma^2_{0E}, \dots, \sigma^2_{IE}$ indicates the different accuracies with which observations are measured at node i.

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Rising Questions:

1. How to estimate the unknown expected state s?
2. Where to allocate the different types of measurements at the node of the network in order to get a precise estimation of the unknown expected state s?

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Estimating the state in the network model

The BLUE for Ls in the network model for state estimation:

$$L\hat{s} = L\left((1_N \otimes \mathbb{S}^{-1/2}\,\mathbb{X})^\top (1_N \otimes \mathbb{S}^{-1/2}\,\mathbb{X})\right)^{-1} (1_N \otimes \mathbb{S}^{-1/2}\,\mathbb{X})^\top \mathbb{I}_{N\times N} \otimes \mathbb{S}^{-1/2}\, Y$$

where the matrix $\mathbb{S}$ is given by

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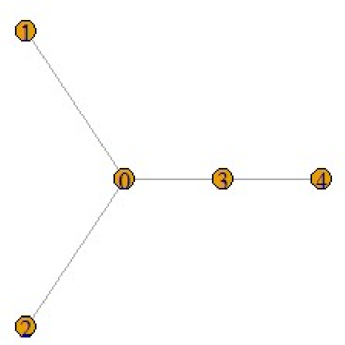

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The covariance matrix of the BLUE

If the influence matrix $\mathbb{X}$ is non-singular, the covariance matrix of $L\hat{s}$ is of the form:

$$\mathrm{Cov}(L\hat{s}) = \frac{1}{N}\left(L\,\mathbb{D}_Z\,L^\top + L\,(\mathbb{X}^\top\,\mathbb{D}_E^{-1}\,\mathbb{X})^{-1}\,L^\top\right).$$

The influence of different measurement types on Cov($L\hat{s}$)

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- $\mathbb{D}_Z$ which describes the covariances of the random states at the different nodes.
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Cov($L\,\hat{s}$) depends on

- $\mathbb{D}_Z$ which describes the covariances of the random states at the different nodes.
- $\mathbb{D}_E$ which describes the variances of the measurement errors at the different nodes.
  - The diagonal entries of $\mathbb{D}_E$ indicate the inaccuracy of the applied measurement procedures at the different nodes.
  - If the applied measurement type is precise at node i, the variance σ_{iE}^2 will be small ($i = 0, \ldots, I$).

Transfer to approximate design theory I

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Set

- $\delta_i := \frac{1}{\sigma_{iE}^2}$, $i = 0, \ldots, I$
- $\delta = (\delta_0, \ldots, \delta_I)$
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We can restrict ourselves to the condition that

$$\delta \in \Delta := \{\delta = (\delta_0, \delta_1, \ldots, \delta_I)^\top \in (0,1)^{I+1};\ \sum_{i=0}^{I} \delta_i = 1\}$$

The approximate design problem

Target: Determine approximate design $\delta = (\delta_0, \ldots, \delta_I) \in \Delta$ such that

$$\tilde{C}(\delta) = \left(L\, \mathbb{D}_Z\, L^\top + L\, (\mathbb{X}^\top\, \mathbb{D}_\delta\, \mathbb{X})^{-1}\, L^\top \right)$$

becomes small in some sense.

Question: Which design criterion should be preferred?

We are interested in estimating $L\, s$.

$\Rightarrow$ Determine A-optimal designs: Determine the design $\delta^* \in \Delta$ that minimizes

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A-optimal design in general network

Theorem

Let the influence matrix $\mathbb{X}$ be non-singular. Then the A-optimal design $\delta^* = (\delta_0^*, \delta_1^*, \ldots, \delta_I^*)$ for estimating $L\,s$, that minimizes

$$tr\Big(L\,(\mathbb{X}^\top\,\mathbb{D}_\delta\,\mathbb{X})^{-1}\,L^\top\Big)$$

is given by

$$\delta_i^* = \frac{\sqrt{v_i}}{\sum_{j=0}^{I}\sqrt{v_j}},$$

$$\text{where } v_i = u_i^\top(\mathbb{X}^{-1})^\top L^\top L\,\mathbb{X}^{-1}u_i \text{ for } \quad i = 0, 1, \ldots, I.$$

Here u_i denotes the $(i+1)$-th unit vector.

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The determination of the A-optimal design reduces to the calculation of the inverse of the influence matrix $\mathbb{X}$.

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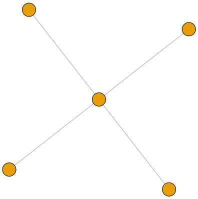

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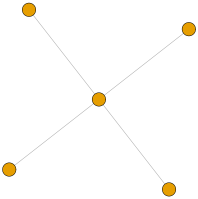

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$\Rightarrow$ A-optimal designs can be calculated analytically.

A-optimal design in star network

Corollary

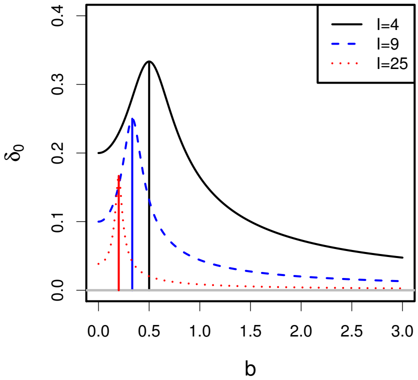
If $b^2 \neq \frac{1}{I} a^2$, then the A-optimal design $\delta^* = (\delta_0^*, \delta_1^*, \dots, \delta_I^*)$ for estimating the expected state vector s in the star network is given by $\delta_0^* = \frac{\sqrt{w}}{I\sqrt{v}+\sqrt{w}}$ and $\delta_i^* = \frac{\sqrt{v}}{I\sqrt{v}+\sqrt{w}}$ for $i = 1, \dots, I$, where

$$w = a^2 + Ib^2 \,,$$

and

$$v = \left(b^2 + \frac{(a^2 - (I-1)b^2)^2}{a^2} + (I-1)\frac{b^4}{a^2} \right) .$$

The weight in the central node of the star network



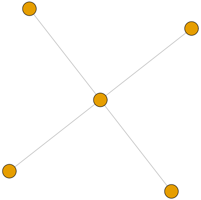
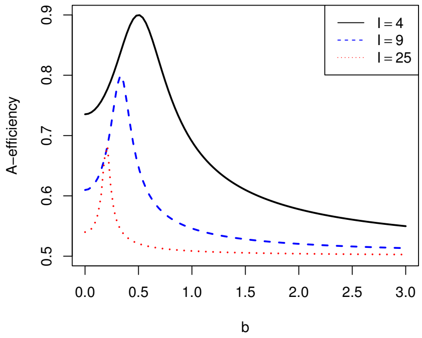


Fig.: Optimal values for δ_0 depending on the quantity b, the influence of the central node 0, for $a = 1$.

A-efficiencies of other designs in the star network

We consider the intuitive design given by $\delta = (0.5, 0.5/I, \dots, 0.5/I)$.



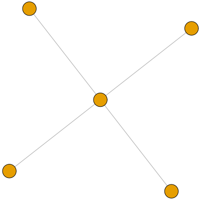
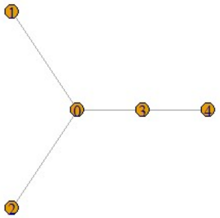


Fig.: A-efficiency of the design which allocates the most precise device in the central node.

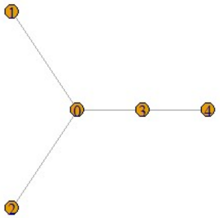
Extension to connected star networks – Shooting star

The network is given by a shooting star, its adjacency matrix is of the form:



$$\mathbb{X} = \begin{pmatrix} a & b & b & b & 0 \\ b & a & 0 & 0 & 0 \\ b & 0 & a & 0 & 0 \\ b & 0 & 0 & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix}$$

Extension to connected star networks – Shooting star

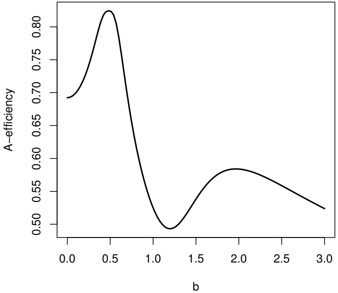


The network is given by a shooting star, its adjacency matrix is of the form:

$$\mathbb{X} = \begin{pmatrix} a & b & b & b & 0 \\ b & a & 0 & 0 & 0 \\ b & 0 & a & 0 & 0 \\ b & 0 & 0 & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix}$$

The A-optimal weights for $b = 0.75$:

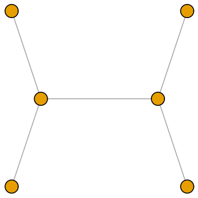
$(0.222, {\color{red}0.175}, {\color{red}0.175}, 0.216, 0.212)$



Extension to connected star networks – Two connected stars

The network is given by two connected stars,
its adjacency matrix is of the form:

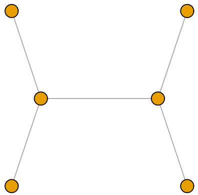
$$\mathbb{X} = \begin{pmatrix} a & b & b & b & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ b & 0 & a & 0 & 0 & 0 \\ b & 0 & 0 & a & b & b \\ 0 & 0 & 0 & b & a & 0 \\ 0 & 0 & 0 & b & 0 & a \end{pmatrix}$$



Extension to connected star networks – Two connected stars

The network is given by two connected stars,
its adjacency matrix is of the form:

$$\mathbb{X} = \begin{pmatrix} a & b & b & b & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ b & 0 & a & 0 & 0 & 0 \\ b & 0 & 0 & a & b & b \\ 0 & 0 & 0 & b & a & 0 \\ 0 & 0 & 0 & b & 0 & a \end{pmatrix}$$



The A-optimal weights for $b = 0.75$:
$(0.174, 0.163, 0.163, 0.174, 0.163, 0.163)$

Conclusion

- A simple random state model can be modelled by using graph theory.
- By a relaxation of the original problem of placing different measurement types of different precision to the nodes of the network, we obtain an approximate design problem.
- The optimization problem can be solved by calculating the inverse of the influence matrix $\mathbb{X}$.
- This design problem can be solved analytically in some specific networks (e.g. star).

Further results and outlook

Further results are available for

- non-simultaneous observations at the different nodes of the network,
- other types of network, for instance the wheel network,
- for the case, where the influence matrix is singular and the complete state vector s is not identifiable anymore.

⇒ *Müller, C.H., Schorning, K.(2023) A-optimal designs for state estimation in networks. Stat Papers.*

Outlook: we currently work on

- more general results on the properties of A-optimal designs for networks: for instance symmetry.
- robust designs w. r. t. the influence matrix $\mathbb{X}$.
- the extension of the model to the case of time-dependent observations.

Thank you!