How to determine the minimal sample size in balanced 3-way ANOVA models where no exact F-test exists

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1 INTRODUCTION

We consider balanced three-way ANOVA models to test the hypothesis that the fixed factor A has no effect. The other factors are fixed or random. For most of these models (including all balanced 1-way and 2-way ANOVA models) an exact F-test exists. Details on the determination of the minimal sample size and on an in-depth structural result can be found in Spangl, et al. (2023).
For the two models

$$A \times \boldsymbol{B} \times \boldsymbol{C} \qquad \text{and} \qquad (A \succ \boldsymbol{B}) \times \boldsymbol{C} \tag{1}$$

(bold letters indicate random factors) an exact F-test does not exist.
We will focus here on the model $(A \succ \boldsymbol{B}) \times \boldsymbol{C}$. Thus, we have the linear model:

$$\boldsymbol{y}_{ijkl} = \mu + \alpha_i + \boldsymbol{\beta}_{j(i)} + \boldsymbol{\gamma}_k + (\boldsymbol{\alpha\gamma})_{ik} + (\boldsymbol{\beta\gamma})_{jk(i)} + \boldsymbol{e}_{ijkl}\,, \quad i = 1, \dots, a, \quad j = 1, \dots, b, \quad k = 1, \dots, c, \quad l = 1, \dots, n\,, \tag{2}$$

with

$$\sigma_y^2 = \mathrm{var}\left(\boldsymbol{y}_{ijkl}\right) = \sigma_{b(a)}^2 + \sigma_c^2 + \sigma_{ac}^2 + \sigma_{bc(a)}^2 + \sigma^2\,. \tag{3}$$

Approximate F-tests can be obtained by Satterthwaite's approximation:

$$\boldsymbol{F}_A = \frac{\boldsymbol{MS}_A + \boldsymbol{MS}_{B\times C(A)}}{\boldsymbol{MS}_{B(A)} + \boldsymbol{MS}_{A\times C}},$$

with corresponding approximate df_1 and df_2.
The approximate F-test involves mean squares to be estimated. To approximate the power of the test, we simulate data such that the null hypothesis is false. The rate of rejections approximates the power of the test. Thus, the power depends on

$$(a,\ b,\ c,\ n,\ \alpha,\ \delta,\ \theta)\ ,$$

where α is the type I risk, $\delta = \alpha_{\max} - \alpha_{\min}$ and $\theta = (\sigma_{b(a)}^2, \sigma_c^2, \sigma_{ac}^2, \sigma_{bc(a)}^2, \sigma^2)$ the tuple of the variance components. A typical empirical power surface is given in Figure 1.

Aim

We aim to determine the minimal sample size of the model mentioned above, given a prespecified power.

Simulations

The calculation of the power is always based on 10 000 simulations. Additionally, we fix $a = 6$, $\alpha = 0.05$, the ratio of the total variance σ_y^2 and δ^2 equal to one, i.e., $\delta^2/\sigma_y^2 = 1$, and $n = 2$. Moreover, the main effects of the factor A are least favorable.

Assumption $n = 2$

The number of replicates n should be kept small ($n = 2$). This assumption is backed by all simulation results.

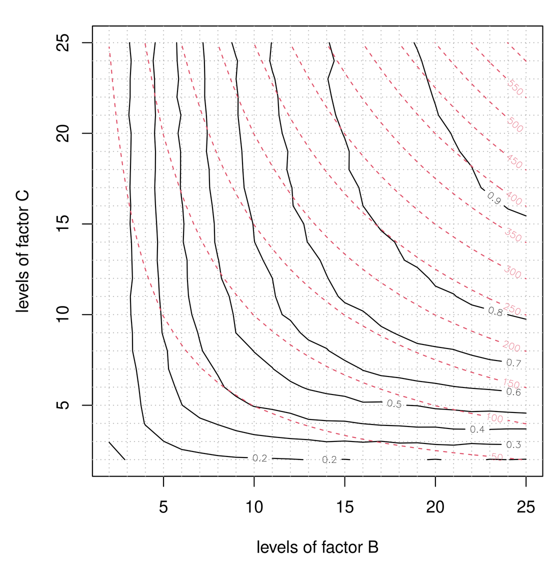

Figure 1: Empirical power surface with variance components $\theta = (10, 5, 5, 0, 5)$.

2 INFLUENCE OF VARIANCE COMPONENTS

Q: Is there a 'pivot' parameter, too, as in ANOVA models where an exact F-test exists (cf. Spangl et al., 2023)?

A: No, the power depends on the variance components θ !

Q: Which variance components are the most influential ones?

A: $\sigma_{b(a)}^2$, σ_{ac}^2 and $\sigma_{bc(a)}^2$.

Method:

Use a 2^5 screening design with variance components as factors.

Q: How do $\sigma_{b(a)}^2$, σ_{ac}^2 and $\sigma_{bc(a)}^2$ influence the empirical power?

A: An example is given in Figure 2.

Method:

We determine the worst combination of active variance components by using a surrogate second-order response surface model that is based on a Box-Behnken design. The special structure of the Box-Behnken design ensures that the used models have similar total variance.
The simplex surface in Figure 2 is spanned by seven design points of the Box-Behnken design. The coloring represents the power values predicted by the second-order response surface model. Green colors represent low power values, red colors high power values.

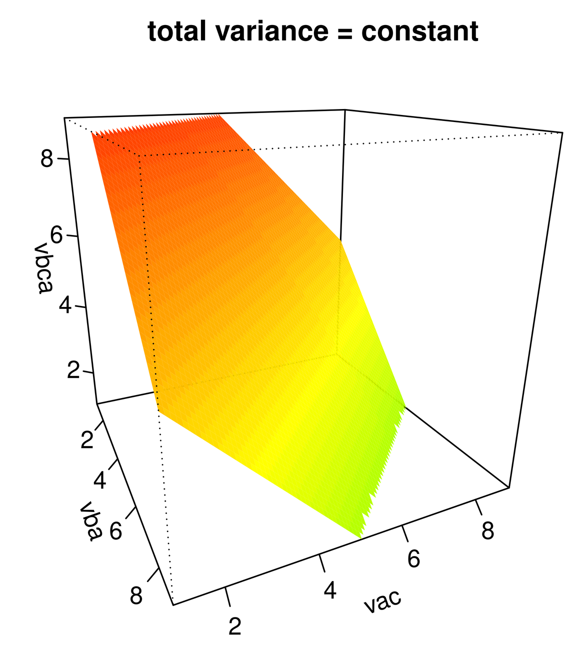

Figure 2: Power (green = low, red = high), for $b = c = 12$, predicted by the second-order response surface model, which varies the three most influential variance components $\sigma_{b(a)}^2$, σ_{ac}^2 and $\sigma_{bc(a)}^2$. The simplex surface is spanned by a subset of seven design points of the Box-Behnken design.

3 SAMPLE SIZE DETERMINATION

We present three different approaches using surrogate models (cf. Figures 3–5).

(i) 'Ray' approach

- Start with simulating a few models, e.g. 6, of equal size and compute the empirical power (blue points in Figure 3).
- Fit a simple model, e.g. using splines, and determine the maximal predicted power.
- Search along this direction for a model with prespecified power (green line in Figure 3).

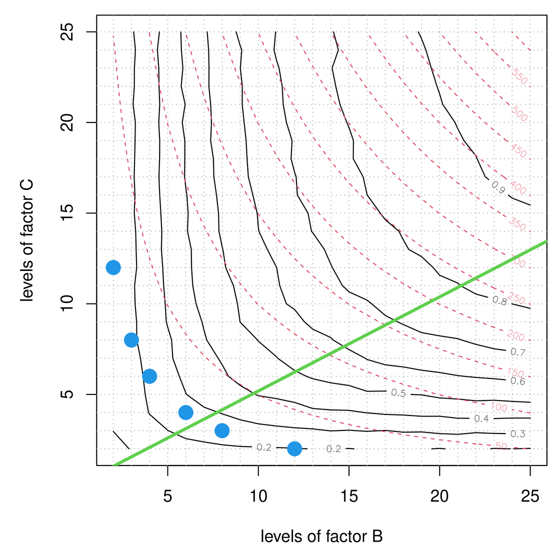

Figure 3: Empirical power surface with variance components $\theta = (10, 5, 5, 0, 5)$. Starting models (blue points). Search direction (green line).

(ii) 'RSM' approach

- Start with a 2^2 screening design with the number of levels b and c as factors and compute the empirical power.
- Use method of steepest ascent.
- Fit a second-order response surface model based on a central composite design. The result is given in Figure 4.

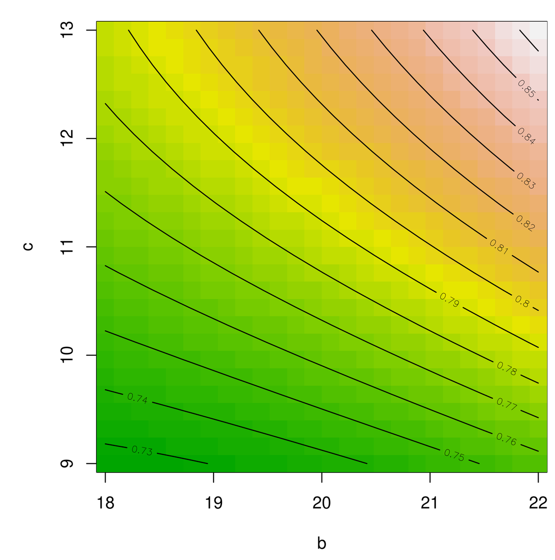

Figure 4: Fitted power surface of second-order response surface model with variance components $\theta = (10, 5, 5, 0, 5)$.

(iii) 'Reciprocal' approach

- Start with simulating the 9 models with $b, c = 3, 4, 5$, and compute the empirical power.
- Fit the model $P_{\text{fit}}(b, c) = \tanh\left(\frac{1}{C_1/b + C_2/c} - C_3\right)$.
- Choose a new model and compute the empirical power. After each additional simulation fit again and repeat.

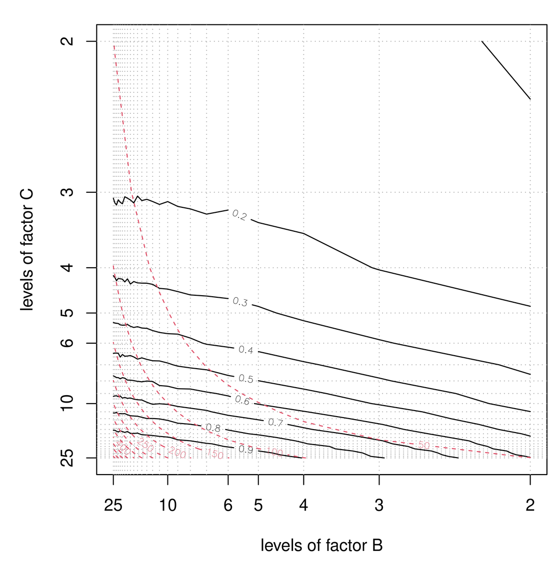

Figure 5: Empirical power surface viewed in reciprocal scaling with variance components $\theta = (1, 5, 9, 5, 5)$.

REFERENCES

Spangl, B., Kaiblinger, N., Ruckdeschel, P., and Rasch, D. (2023). Minimal sample size in balanced ANOVA models of crossed, nested, and mixed classifications. *Communications in Statistics – Theory and Methods*, **52(6)**, 1728 – 1743.